Raisa Koval', On a Functional Equation with a Group Isotopy Property, *Bul. Acad. Ştiinţe Repub. Mold. Mat.*, 2005, Number 2, 65–71

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(48), 2005, Pages 65–71
ISSN 1024–7696

# On a Functional Equation with a Group Isotopy Property

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**Abstract.** The set of all solutions of functional equation $F_1(F_2(z;x);F_3(y;z)) = F_4(F_5(x;y);x)$ on quasigroup operations of an arbitrary fixed set are found. The result implies W. Dudek's theorem [1], which presents the operation in a quasigroup satisfying the identity $xy \cdot x = zx \cdot yz$.

**Mathematics subject classification:** 20N15, 20N05.
**Keywords and phrases:** Quasigroup, functional equation, group isotope, identity.

Functional equations play a specific role in the quasigroup theory. V.D. Belousov was the first who opened the new field. He announced in [2] the fact that later became known as the Four quasigroup theorem. It was strengthened, complitely proved and published in [3] two years later. The theorem received general acceptance and wide application. He was also the first who applied the following theorem: *if a quasigroup satisfies an uncancellable balanced identity, then it is isotopic to a group* [4].

In [5] such identities are called identifies with *group isotopy property* (gip) and a functional equation has a *(full) group isotopy property* if some (correspondingly, any) component of every solution of the equation is isotopic to a group.

The class of all general quadratic functional identities with gip was described in [6,7]. In [8] a complete classification was presented up to parastrophic equivalency of all general parastrophic uncancellable quadratic functional equations having $n$ objective variables for $n = 3$, 4, 5. The results imply that there exists one equation when $n = 3$ (general associativity); two equations when $n = 4$ (general mediality and general pseudomediality); and four equations when $n = 5$. These are the only equations having a full gip for all $n = 3, 4, 5$.

Here we give a clear proof of the results announced in [9]. Namely, we consider a general functional equation which is not quadratic, but has a gip:

$$F_1(F_2(z;x);F_3(y;z)) = F_4(F_5(x;y);x). \qquad (1)$$

This equation corresponds to the identity $xy \cdot x = zx \cdot yz$ considered by W.Dudek [1]. We obtained some of his results as consequences of our main theorem.

**Theorem 1** (Four quasigroup theorem,[10])**.** *The set of all solutions of the general associativity equation*

$$F_1\left(F_2(x;y), z\right) = F_3\left(x, F_4(y;z)\right) \qquad (2)$$

*on the set of all binary quasigroup operations of an arbitrary fixed set $Q$ is described by:*

$$F_1(t,z)=\mu t+\gamma z,\qquad F_2(x;y)=\mu^{-1}\big(\alpha x+\beta y\big),$$

$$F_3(x,u)=\alpha x+\nu u,\quad F_4(y;z)=\nu^{-1}\big(\beta y+\gamma z\big),$$

*where $(Q;+)$ is a group, $\mu$, $\nu$, $\alpha$, $\beta$, $\gamma$ are substitutions on the set $Q$.*

The following assertion easily follows from the Four quasigroup theorem.

**Corollary 2.** *The set of all solutions of the functional equation*

$$F_1\left(F_2(x;y),z\right)=F_1\left(x,F_3(y;z)\right) \tag{3}$$

*on the set of all binary quasigroup operations of a set $Q$ are described by the following equalities:*

$$F_1(x;y)=\alpha x+\gamma y,\quad F_2(x;y)=\alpha^{-1}(\alpha x+\beta y),\quad F_3(x;y)=\gamma^{-1}(\beta x+\gamma y), \tag{4}$$

*where $(Q;+)$ is a group, $\alpha$, $\beta$, $\gamma$ are substitutions of the set $Q$.*

**Proof.** Let a triple $(f_1,f_2,f_3)$ of quasigroup operations, defined on an arbitrary fixed set $Q$, be a solution of (3), then the tuple $(f_1,f_2,f_1,f_3)$ is a solution of (2), and then

$$\begin{aligned} f_1(t,z)&=\mu t+\gamma_0 z,\quad f_2(x;y)=\mu^{-1}\big(\alpha x+\beta_0 y\big),\\ f_1(x,u)&=\alpha x+\gamma u,\quad f_3(y;z)=\gamma^{-1}\big(\beta_0 y+\gamma_0 z\big)\end{aligned} \tag{5}$$

for a group $(Q;+)$ and substitutions $\alpha$, $\beta_0$, $\gamma_0$, $\gamma$, $\mu$ of the set $Q$. So

$$\mu t+\gamma_0 z=\alpha t+\gamma z \tag{6}$$

for all $t$, $z\in Q$. Let 0 denote the neutral element of the group $(Q;+)$. Putting $z:=\gamma_0^{-1}0$ and $t:=\alpha^{-1}0$ we obtain two equalities:

$$\mu(t)=\alpha t+\gamma(\gamma_0^{-1}0),\qquad \gamma(z)=\mu(\alpha^{-1}0)+\gamma_0 z.$$

Substitute them into (6):

$$\alpha t+\gamma(\gamma_0^{-1}0)+\gamma_0 z=\alpha t+\mu(\alpha^{-1}0)+\gamma_0 z.$$

It implies that $\gamma(\gamma_0^{-1}0)=\mu(\alpha^{-1}0)=:a$, so $\mu=R_a\alpha$, $\gamma=L_a\gamma_0$. Denoting $\beta:=R_a^{-1}\beta_0$, we obtain the relations (4). □

The next statement is evident, but one can find its proof in [11] or in [12]. We recall, a substitution $\alpha$ of a group $(Q;+)$ is said to be *unitary*, if $\alpha 0=0$, where 0 denotes the neutral element of the group.

**Lemma 3.** *Let substitutions $\alpha$, $\beta$, $\gamma$, $\delta$, $\mu$ of a group $(Q;+)$ satisfy the identity*

$$\alpha(\beta x + \gamma y) = \delta u + \mu v,$$

*then $\alpha$ is an automorphism (antiautomorphism) of the group $(Q;+)$ if $\alpha$ is unitary and $u = x$, $v = y$ (corresponding $u = y$, $v = x$).*

Below we will follow the notations

$$L_i x := f_i(a;x), \qquad R_i x := f_i(x;a), \qquad i = 1,2,3. \tag{7}$$

**Theorem 4.** *A tuple $(f_1, \ldots, f_5)$ of quasigroup operations defined on a set $Q$ is a solution of the functional equation (1) iff there exist a group operation $(+)$ and substitutions $\alpha$, $\beta$, $\gamma$, $\delta$, $\mu$ of the set $Q$ such that*

$$\begin{aligned} f_1(x;y) &= \alpha x + \gamma y, & f_2(x;y) &= \alpha^{-1}(\mu y - \beta x), \\ f_3(x;y) &= \gamma^{-1}(\beta y + \delta x), & f_5(x;y) &= f_4^{\ell}(\mu x + \delta y; y) \end{aligned} \tag{8}$$

*for a quasigroup operation $f_4$ being orthogonal to the group isotope $(\circ)$, where $x \circ y := \mu x + \delta y$.*

**Proof.** Let a tuple $(f_1, \ldots, f_5)$ of quasigroup operations defined on an arbitrary fixed set $Q$ be a solution of the functional equation (1), i.e. the equality

$$f_1(f_2(z;x); f_3(y;z)) = f_4(f_5(x;y);x) \tag{9}$$

holds for all $x$, $y$, $z \in Q$. Let $a$ be an element of the set $Q$ and let $z := a$, then we get

$$f_4(f_5(x;y);x) = f_1(L_2 x; R_3 y).$$

Comparing this equality with (9) we come to

$$f_1(L_2 x; R_3 y) = f_1(f_2(z;x); f_3(y;z)).$$

To transform it into a general associativity-like equation we replace $y$ with $R_3^{-1} y$, $x$ with $f_2^r(z;x)$:

$$f_1(L_2 f_2^r(z;x); y) = f_1(x; f_3(R_3^{-1} y; z))$$

and replace $f_3$ and $f_2^r$ with their transposed ones:

$$f_1(L_2 f_2^{r*}(x;z); y) = f_1(x; f_3^*(z; R_3^{-1} y)).$$

Since $f^{r*} = f^{rr\ell r} = f^{\ell r}$, then, designating

$$f_2'(t;z) := L_2 f_2^{lr}(t;z), \quad f_3'(z;y) := f_3^*(z; R_3^{-1} y), \tag{10}$$

we receive the identical equality

$$f_1(f_2'(x;y); z) = f_1(x; f_3'(y;z)),$$

which means that the tuple $(f_1, f_2', f_1, f_3')$ of the operations is a solution of the associativity functional equation (3). Corollary 2 implies the existence of a group $(Q;+)$ and substitutions $\alpha$, $\beta$, $\gamma$ such that

$$f_1(x;y) = \alpha x + \gamma y, \quad f_2'(x;y) = \alpha^{-1}(\alpha x + \beta y), \quad f_3'(x;y) = \gamma^{-1}(\beta x + \gamma y) \tag{11}$$

come true. Let us find the operations $f_1$, $f_2$, $f_3$. The first equality of (11) coincides with (8) for the operation $f_1$. Taking into account the third equality of (11) and the notation (10) we have

$$f_3^*(x; R_3^{-1}y) = \gamma^{-1}(\beta x + \gamma y)$$

and, consequently,

$$f_3(y;x) = \gamma^{-1}(\beta x + \gamma R_3 y).$$

Designating $\delta := \gamma R_3$ we receive the relation (8) for the operation $f_3$.

Again, from the equalities (10) and (11) we have

$$L_2 f_2^{\ell r}(x;y) = \alpha^{-1}(\alpha x + \beta y).$$

Apply the substitution $L_2^{-1}$ to the both sides of the last equality:

$$f_2^{\ell r}(x;y) = (\alpha L_2)^{-1}(\alpha x + \beta y).$$

Since $f_2^{\ell r} = (f_2^{\ell})^r$, according to the definition of the left and right quasigroup divisions we obtain

$$f_2(y; (\alpha L_2)^{-1}(\alpha x + \beta y)) = x.$$

Let us designate $\mu := \alpha L_2$ and $t := \mu^{-1}(\alpha x + \beta y)$, then $x = \alpha^{-1}(\mu t - \beta y)$. Thus, the operation $f_2$ has decopmosition (8).

Using the received expressions (8) for the operations $f_1$, $f_2$, $f_3$, calculate the left part of the equality (9):

$$\begin{aligned} f_1(f_2(z;x); f_3(y;z)) &= \alpha f_2(z;x) + \gamma f_3(y;z) = \\ &= \mu x - \beta z + \beta z + \delta y = \mu x + \delta y. \end{aligned} \tag{12}$$

Taking into consideration (12), the equality (9) can be written as:

$$f_4(f_5(x;y);y) = \mu x + \delta y.$$

It is equivalent to (8) for the operation $f_5$. By virture of the fact that $f_5$ is a quasigroup operation, the operation $f_4$ is orthogonal to $(\circ)$, where $x \circ y := \mu x + \delta y$.

Vise versa, let $(Q;+)$ be a group, $\alpha$, $\beta$, $\gamma$, $\delta$, $\mu$ be substitutions of $Q$ and $f_4$ be a quasigroup operation being orthogonal to $(\circ)$. The operations $f_1$, $f_2$, $f_3$, defined by (4), are group isotopes therefore they are quasigroups. Orthogonality of $f_4$ and $(\circ)$ implies that $f_5$ is a quasigroup operation too.

Let us show that the tuple $(f_1, \dots, f_5)$ of quasigroup operations defined by (8) on $Q$ is a solution of the functional equation (1), i.e. (9) holds. The relation (12) gives

$$f_1(f_2(z;x);f_3(y;z)) = \mu x + \delta y.$$

Consider the right side of (9):

$$f_4(f_5(x;y);x) = f_4(f_4^{\ell}(\mu x + \delta y;y);y) = \mu x + \delta y.$$

The left sides are equal as the right sides of these equalities are equal. □

As a consequence we obtain the W. Dudek's result from [1], which we give in another form.

**Corollary 5.** *A groupoid* $(Q;\cdot)$ *satisfies the identity*

$$xy \cdot x = zx \cdot yz \tag{13}$$

*if and only if* $x \cdot y = \varphi x + (\varepsilon - \varphi)y$ *for some automorphism* $\varphi$ *of a commutative group* $(Q;+)$ *such as* $\varepsilon - \varphi$ *is a substitution of the set* $Q$ *and the following relation is true*

$$2\varphi^2 - 2\varphi + \varepsilon = 0. \tag{14}$$

**Proof.** Fulfilment of the identity (13) means that the tuple $(\cdot;\cdot;\cdot;\cdot;\cdot)$ is a solution of the functional equation (1), so the relations (8) are true, where every of the operations $f_1, \dots, f_5$ coincides with $(\cdot)$.

According to Lemma 3 the equalities $f_2 = f_1$ and $f_3 = f_2$ imply that $\alpha$ and $\gamma$ are alinear transformations of the group $(Q;+)$, i.e.

$$\alpha = R_b\varphi, \qquad \gamma = L_c\psi \tag{15}$$

for some elements $b$, $c \in Q$ and antiautomorphisms $\varphi$, $\psi$ of the group $(Q;+)$.

Let 0 denote the neutral element of the group $(Q;+)$. According to (13) the equality $00 \cdot 0 = 00 \cdot 00$ it true. By (8) and (15) it can be written as

$$\varphi(\varphi 0 + a + b + \psi 0) + a + b + \psi 0 = \varphi(\varphi 0 + a + b + \psi 0) + a + b + \psi(\varphi 0 + a + b + \psi 0).$$

Taking into account that $\varphi 0 = \psi 0 = 0$, we have

$$\varphi(a+b) + a + b = \varphi(a+b) + a + b + \psi(a+b).$$

Therefore $a + b = 0$, and the operation $(\cdot)$ has the decomposition

$$x \cdot y = \varphi x + \psi y. \tag{16}$$

Because of this the identity (13) can be written as

$$\varphi\psi y + \varphi^2 x + \psi x = \varphi\psi x + \varphi^2 z + \psi^2 z + \psi\varphi y. \tag{17}$$

Putting $x = y = 0$, $x = z = 0$, $y = z = 0$, we have the following equalities

$$\varphi^2 = I\psi^2, \qquad \varphi\psi = \psi\varphi, \qquad \varphi^2 + \psi = \varphi\psi. \tag{18}$$

The first equality implies that $I = \varphi^2\psi^{-2}$. Since $\varphi$ and $\psi$ are antiautomorphisms, then $\varphi^2$ and $\psi^{-2}$ are automorphisms of the group $(Q;+)$. So $I$ is its automorphisms. Consequently, the group is commutative and $\varphi$, $\psi$ are its automorphisms too.

Replace $\varphi^2$ with $I\psi^2$ and $\varphi\psi$ with $\psi\varphi$ in the third equality of (18):

$$\psi - \psi^2 = \psi\varphi.$$

It implies $\psi = \varepsilon - \varphi$, so $\varepsilon - \varphi$ is a substitution of the set $Q$. Summing up the obtained relations we have

$$0 = \varphi^2 + \psi^2 = \varphi^2 + (\varepsilon - \varphi)^2 = \varphi^2 + \varepsilon^2 - 2\varphi + \varphi^2 = 2\varphi^2 - 2\varphi + \varepsilon,$$

i.e. (14) is true.

Vice versa, let $(Q;\cdot)$ be a quasigroup and $x \cdot y = \varphi x + (\varepsilon - \varphi)y$ for some automorphism $\varphi$ of a commutative group $(Q;+)$ such that $\varepsilon - \varphi$ is a substitution of the set $Q$ and equality (14) is true. Then the last equality immediately implies $\varphi^2 - \varphi + \varepsilon = \varphi - \varphi^2$ and so we have

$$\begin{aligned} xy \cdot x &= \varphi(\varphi x + (\varepsilon - \varphi)y) + (\varepsilon - \varphi)x = \varphi^2 x + \varphi(\varepsilon - \varphi)y + (\varepsilon - \varphi)x = \\ &= (\varphi^2 - \varphi + \varepsilon)x + (\varphi - \varphi^2)y = (\varphi - \varphi^2)x + (\varphi - \varphi^2)y. \end{aligned}$$

The right part of (13) can be calculated by the same way:

$$\begin{aligned} zx \cdot yz &= \varphi(\varphi z + (\varepsilon - \varphi)x) + (\varepsilon - \varphi)(\varphi y + (\varepsilon - \varphi)z) = \\ &= \varphi^2 z + \varphi(\varepsilon - \varphi)x + \varphi(\varepsilon - \varphi)y + (\varepsilon - \varphi)^2 z = \\ &= (2\varphi^2 - 2\varphi + \varepsilon)z + (\varphi - \varphi^2)x + (\varphi - \varphi^2)y = (\varphi - \varphi^2)x + (\varphi - \varphi^2)y. \end{aligned}$$

Since the right sides are equal, then the left sides are equal as well. □

## References


[1] Dudek W.A. *Quadratical quasiqroups*. Quasigroups and Related Systems, 1997, **4**, p. 9–13.

[2] Belousov V.D. *Associative system of quasigroups*. Uspekhi Mat. nauk, 1958, **13**, N 3(81), p .243 (In Russian).

[3] Aczél J., Belousov V.D., Hosszú M. *Generalized associativity and bisymmetry on quasigroups*. Acta. Math. Acad. Sci. Hung., 1960, **11/1-2**, p. 127–136.

[4] Belousov V.D. *One theorem on balanced equations*. Kvasigruppy i latinskiye kvadraty: Mat. Issled., vyp. 71. Kishinev, Stiintsa, 1983, p. 22–24 (In Russian)

[5] Sokhatsky F. *Group Isotopy Property Functional Equations*. Fifth International Algebraic Conference in Ukraine, Odessa, 2005 July, 20-27.

[6] Krapež A. *Stricly quadratic functional equations on quasigroups I.* Publ. Inst. Math. (Beograd) (N.S.), 1981, **29(43)**, p. 125–138.

[7] Krstić S. *Kvadratni kvazigrupni identiteti.* PhD thesis, University of Belgrade, 1985.

[8] Sokhatsky F.M. *A classification of functional equations on quasigroups.* Ukrainian Mathematical Journal. 2004, **56**, N 4, p. 1259-1266.

[9] Koval R. *On Group Isotopy Property Functional Equation.* Fifth International Algebraic Conference in Ukraine, Odesa, 2005 July, 20-27.

[10] Belousov V.D. *$N$-ary quasigroups.* Kishinev, Shtiintsa, 1972, p. 227.

[11] Belousov V.D. *Foundations of the theory of guasigroups and loops.* Moscva, Nauka, 1967 (In Russian).

[12] Sokhatskii F.N. *On Isotopes of Groups. I.* Ukrainian Mathematical Journal, 1995, **47**, N 10, p. 1585–1598.

[13] Koval R.E. *A classification a quadratic parascophically uncancelled functional equations with five subject variables on quasigroups.* Ukrainian Mathematical Journal, 2005, **57**, N 8, p. 1058–1068.

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*Received May 23, 2005*